Mathematical Structuralism Today

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Abstract

Two topics figure prominently in recent discussions of mathematical structuralism: challenges to the purported metaphysical insight provided by *sui generis* structuralism and the significance of category theory for understanding and articulating mathematical structuralism. This paper presents an overview of central themes related to these topics.

Whether one considers Richard Dedekind's discussion of simple infinite systems in his treatment of arithmetic, the development of abstract algebra, or 19th and 20th Century geometry, it is hard to ignore the role of structure and/or sameness of structure in mathematics. Considering this data, it is easy to see why many endorse the basic structuralist[1] thesis that "mathematics investigates structure". Unfortunately, structuralists mean quite different things by this thesis.

An important distinction is between those who acknowledge abstract (rather than merely concrete) structures and those who do not; label the former *non-eliminative structuralists* and the latter *eliminative structuralists*. The central theses of *non-eliminative structuralism* (NES) are, roughly, a) pure mathematics is the investigation of *abstract* mathematical structures, and b) only the structural properties of objects matter for mathematics. NESists take these theses to express a fundamental insight into the nature of mathematical objects, namely, that such objects are *positions* or *places*[2] in abstract mathematical structures. *Eliminative structuralism* (ES) rejects abstract mathematical structures and, consequently, that the nature of mathematical objects is exhausted by their being places in such structures. Roughly, it takes mathematics to be the investigation of *concrete* (or neutral)[3] structures, where concrete structures are those instantiated in the ESists' background ontology. Some ESists work with a nominalistically acceptable background ontology,[4] while others supplement this with some non-structurally construed mathematical objects.[5] Since ES will play a minimal role in this paper, we shall forgo any further explication of it.[6]

Two topics, debated in largely non-overlapping literature, figure prominently in recent discussions of structuralism: challenges to the purported metaphysical insight provided by NES—in particular, the type of NES committed to *sui generis* mathematical objects, i.e., *sui generis* structuralism (SGS)[7]—and the significance of category theory for understanding and articulating structuralism. These topics will be the focus of this paper. Specifically, in §I, I shall elucidate SGS and its purported metaphysical insight, concentrating on Michael Resnik's and Stewart Shapiro's work in the 1980s and 1990s, in §II, I shall discuss recent challenges to this purported insight, and, in §III, I shall discuss the debate surrounding the significance of category theory for structuralism.

Before we begin, however, let me note that structuralists conceive of structure in two distinct ways. Some, prompted by abstract algebra, take a top-down conception of structure, whereby structure is characterized and revealed by structure-preserving mappings, to be fundamental.

Others subscribe to a bottom-up conception grounded in Zermelo-Fraenkel style foundations.[8] Lamentably, the top-down conception of structure has had little impact on discussions of SGS and its purported metaphysical insight.[9] Consequently, I shall postpone discussion of this conception until §III.

In §I and §II I shall exclusively utilize a *bottom-up* conception of structure and structural property. Shapiro's characterization of the former is helpful:

> I define a *system* to be a collection of objects with certain relations. … A *structure* is the abstract form of a system, highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system. ('Philosophy of Mathematics' 73–4)

Accordingly, a *structural property* is one that can be obtained by ignoring any features of the objects of a system "that do not affect how they relate to other objects in the system". Thus, a property of a system, S, is structural iff it is shared by every system with the same structure as S.

To illustrate these notions and their relation to mathematics, consider the system of Arabic numerals, i.e., 0, 1, 2, 3, …, and the system of Zermelo numerals, i.e., Ø, {Ø}, {{Ø}}, {{{Ø}}}, …. There are many features that these systems do not share, e.g., they have different initial elements. Yet there are several features that they do share, e.g., they each have a distinguished initial element, each element other than the initial element is the successor of another element, and the successor relation on them satisfies the induction principle. These three are structural properties of these systems. The abstract form of these systems, and any others that have a distinguished initial object and a successor relation that satisfies the induction principle, is the natural-number structure.

I. Sui Generis *Structuralism and its Metaphysical Insight*

Resnik and Shapiro defend different SGSs and, consequently, offer different accounts of SGS's metaphysical insight. Yet, despite differences, there are three types of theses to which both are (or were) committed.[10] The first ascribes a type of dependence to the places of structures:

> *Dependence*: Mathematical objects are dependent on the structure (and/or the other objects of the structure) to which they belong.[11]

Before formulating the other two types of theses, let us review some expressions of them:

> Mathematics is concerned with structures involving mathematical objects and not with the 'internal' nature of the objects themselves. (Resnik, 'Ontology and Reference' 529)

> In mathematics … we do not have objects with an 'internal' composition arranged in structures, we have only structures. The objects of mathematics … are structureless points or positions in structures. As positions in structures, they have no identity or features outside of a structure. (Resnik, 'Ontology and Reference' 530)

> Mathematical objects … have their identities determined by their relationships to other positions in the structure to which they belong. (Resnik, 'Epistemology' 95)

> … the essence of a natural number is the relations it has with other natural numbers. (Shapiro, 'Philosophy of Mathematics' 5)

> [There] is no more to the individual numbers 'in themselves' than the relations they bear to each other. (Shapiro, 'Philosophy of Mathematics' 73)

> Mathematical objects are incomplete in the sense that we have no answers within or without mathematics to questions whether the objects one mathematical theory discusses are identical to those another treats. (Resnik, 'Science of Patterns' 90)

> … it makes no sense to pursue the identity between a place in the natural-number structure and some other object, expecting there to be a fact of the matter. Identity between natural numbers is determinate; identity between numbers and other sorts of objects is not. (Shapiro, 'Philosophy of Mathematics' 79)

These passages make two (complementary) claims about the metaphysics of mathematical objects: i) it is inherently structural, and ii) there is no more to it than is ascribed in i. Further, both claims are made using a variety of metaphysical notions, including 'internal nature', 'internal composition', 'identity', 'feature', and 'essence'. As these notions are complex and inexact, it is impossible to precisely delineate the relationships between various formulations of i and ii. Yet, drawing on the two underlying themes—'property' (P) and 'identity' (ID)—I offer two formulations of each:

> *P-Nature*: The metaphysical nature of mathematical objects is determined by their structural properties,

> *ID-Nature*: The identity of mathematical objects is determined by their structural properties,

> *P-Incompleteness*: Mathematical objects are incomplete in the sense that they have no non-structural (and/or intrinsic) properties,

and

> *ID-Incompleteness*: Mathematical objects are incomplete in the sense that the only determinate facts concerning their identity are those that concern their identity with the places of the structure to which they belong.

These four theses, together with Dependence, express, at least roughly, the core insights that distinguish SGS from more traditional realist accounts of mathematics.[12]

II. Metaphysical Insight?

The primary challenges to SGS concern the truth (or intelligibility) of these five theses. Yet, if SGSists reject these theses, then SGS is indistinguishable from traditional realist accounts of mathematics. Consequently, according to its opponents, "either [SGS is] bad news or it's old news" (MacBride, 'Structuralism Reconsidered' 584).

To see the problem with these theses, consider first P-Incompleteness. The literature[13] is replete with plausible counterexamples. Generally, these fit into three categories: intentional properties (e.g., being my favorite number), properties of application (e.g., being the number of fundamental forces), and categorical properties (e.g., being abstract). Indeed, as John Burgess observes, some formulations of P-Incompleteness are inconsistent, for the property of having only structural properties is not itself a structural property.[14]

Additionally, the third class of counterexamples to P-Incompleteness—if not also the second[15]—presents a problem for P-Nature. While it is plausible that the intentional properties of mathematical objects are not part of their nature, it is highly implausible that the same is true of such fundamental categorical properties as 'being abstract' and 'being independent of human beings'. Yet, such categorical properties are not structural, and it is unlikely that they are in any sense 'determined' by the structural properties of mathematical objects.

Next, consider ID-Incompleteness. As Fraser MacBride ('Structuralism Reconsidered') emphasizes, this thesis is, at least *prima facie*, incompatible with mathematical practice, which at least seems to be[16] replete with cross-structural identifications.[17] For example, it is standard practice in mathematics to identify the natural numbers with a particular collection of integers and a particular ω-sequence of real numbers. The same is true of the real numbers and a particular collection of complex numbers. Indeed, as Georg Kreisel observes, such identifications are essential to mathematics, for:

> … very often the mathematical properties of a domain *D* only become graspable when one embeds *D* in a larger domain *D´*. Examples: (1) *D* integers, *D´* complex plane; use of analytic number theory. (2) *D* integers, *D´ p*-adic numbers; use of *p*-adic analysis. (166)

A further point emphasized by MacBride ('Structuralism Reconsidered') is that to the extent that ID-Incompleteness is intended to provide an insight into mathematical objects specifically, rather than objects in general, it fails. Consider, for example, Frege's (§56) infamous concern about whether the natural number 2 is Julius Caesar. According to ID-Incompleteness, it is indeterminate whether these two objects are identical. Yet, if this identity is indeterminate, then not only the number 2, but also Julius Caesar, is incomplete. So, ID-Incompleteness implies that *all* objects are incomplete in the sense that there are no determinate facts concerning their identity with certain other objects.

We turn next to ID-Nature. One might suppose that if the identity of mathematical objects is determined by their structural properties, then mathematical objects that share their structural properties are identical. The identity of structurally indistinguishable mathematical objects is also a consequence of Identity of Indiscernibles conjoined with a version of P-Incompleteness. Identity of Indiscernibles implies that it is possible to provide necessary and sufficient conditions for the identity of objects in property-theoretic terms, while the appropriate version of P-Incompleteness limits SGS to invoking only structural properties. Unfortunately for SGSists, it is a straightforward fact that there are distinct mathematical objects that share their structural properties. An *automorphism* on a structure is a one-to-one and onto function from the places of the structure to themselves that preserves structural properties. A structure is *rigid* iff its only automorphism is the identity function. Mathematics is replete with non-rigid structures. The most commonly cited example is the complex numbers; each complex number a + b*i* shares all of its structural properties with its conjugate a – b*i*. Other examples include the group of integers under addition and Euclidean spaces. All non-rigid structures contain distinct places that share their structural properties.

The problem that non-rigid structures present for SGS was raised by Burgess and given its canonical formulation by Jukka Keränen (‘Identity Problem’). These works sparked an ongoing, lively discussion about the legitimacy of applying Identity of Indiscernibles and related metaphysical principles to mathematical objects. This discussion seems to have, as Shapiro observes, “reached a metaphysical standoff” (‘Tale of *i* and -*i*’ 289). Authors[18] on one side maintain, frequently on the basis of intuitions concerning spatio-temporal objects, that such metaphysical principles are universal and so apply to mathematical objects. Those on the other side[19] maintain that these principles are not universal and so need not pertain to mathematical objects. Frequently, representatives of the latter group support their contention by observing the conflict between such principles and SGS’s interpretation of mathematics. At times, compromise positions have been suggested, yet none has been accepted. For example, Ladyman (‘Mathematical Structuralism’) proposed that Identity of Indiscernibles be read as merely requiring that distinct objects be 'weakly discernible'.[20] He withdrew this proposal in 'Identity and Diversity'.[21]

The primary proponent of the final challenge to SGS's purported insight that we shall consider is Hellman ('Three Varieties', 'Structuralism'). He summarizes this challenge as follows:

> If we do not appeal to the relata of a structure as somehow independently given …, but as determined by structural relations … what have we to go on in specifying structural relations other than the axioms … themselves? But … the axioms don't distinguish any particular realization from among the many systems that satisfy them. What, for example, can it mean to speak of *"the ordering"* of *"the natural numbers"* as objects of a [*sui generis*] structure unless we already understand what these numbers are *apart from their mere position in that ordering*? Surely the notion of *"next"* makes no sense except *relative to an ordering or function or arrangement of some sort*…Thus the notion of a [*sui generis*] structure seems to invoke a vicious circularity: such a structure is supposed to consist of *purely structural relations among purely structural objects*, *but understanding either of these requires already understanding the other*. ('Structuralism' 545)

By focusing on this "vicious circularity", Hellman questions the intelligibility of SGS. MacBride ('Numerical Diversity' 67) expresses, but does not endorse, a related metaphysical point:

> In order for objects to be eligible to serve as the terms of a … relation they must be independently constituted as numerically diverse. Speaking figuratively, they must be numerically diverse 'before' the relation can obtain; if they are not constituted independently of the obtaining of a … relation then there are simply no items available for the relation in question to obtain between.

Underwriting MacBride's point is a principle that Øystein Linnebo ('Structuralism') labels 'Universal Downward Dependence' (UDD), i.e., all structures depend on their constituent objects. This principle, Linnebo suggests, can be obtained by generalizing from the spatio-temporal, "where … *downwards dependence* appears to dominate" (68), (e.g., medium-sized physical objects depend on their atomic constituents). UDD is in conflict with Dependence, which maintains that mathematical objects are upwardly-dependent on their structures. Thus, once again, we find a metaphysical standoff between proponents and opponents of SGS; its proponents insist that an upward dependence of an object on a structure is possible, while its opponents insist that all dependence between structures and their constituents is downwards.

Above, we examined significant challenges to all five of the theses central to SGS's purported metaphysical insight. In light of this, one might be tempted to deem SGS unsuccessful. Yet there is an alternative, perhaps better, conclusion to draw: SGS still has not been formulated and framed in a genuinely illuminating way. Support for this contention includes the diversity in formulation of SGS's core insights, Parsons' repeated remarks about the difficulties involved in stating NES (e.g., ('Structuralist View' 303) and ('Mathematical Thought' 42)), numerous changes that SGSists—particularly Shapiro ('Structure and Identity', 'Tale of *i* and –*i*')—have made to their views, and the large number who acknowledge SGS's intuitive appeal. Unfortunately, it is not the aim of this paper to explore an alternative formulation of SGS.[22]

III. Structuralism and Category Theory

Our discussions so far have utilized a model-theoretic or bottom-up conception of structure. A survey of mathematics reveals the prevalence of a *top-down* conception whereby structure is characterized and revealed by structure-preserving mappings.[23] For example, topological structure is that which is characterized and revealed by continuous mappings. The pervasiveness of this top-down conception of structure is easily explained; it is both more flexible than its bottom-up alternative and has been extremely effective in producing powerful tools for proving theorems.[24]

Category theory (CT) provides a theoretical framework for this top-down conception of structure. As Steve Awodey ('Structuralism' 212) explains:

> A category provides a way of characterizing and describing mathematical structure of a given kind, namely in terms of preservation thereof by mappings between mathematical objects bearing the structure in question. A category may be viewed as consisting of objects bearing a certain kind of structure together with mappings between such objects preserving that structure. For example, topological spaces and continuous mappings between them form a category.

For our purposes, three features of CT are noteworthy: the independence of a categorical characterization of some structure from its initial means of specification, which partially explains the label 'top-down', CT's ability to provide precise definitions of 'same structure' and 'structural property' by means of its general notion of isomorphism, and the fact that

> … the only properties which a given object in a given category may have, *qua* object in that category, are structural ones. … Thus doing mathematics 'arrow-theoretically'[25] automatically provides a structural approach [to mathematics]. (Awodey, 'Structuralism' 214-5)

So, at least *prima facie*, CT provides an ideal framework for articulating structuralism.

Recognizing this, Awodey ('Structuralism' 235-6) suggested that

> … [structuralism] be pursued using a technical apparatus other than that developed by logical atomists since Frege, one with a mathematical heritage sufficiently substantial, and

> mathematical applications sufficiently uniform, to render significant a view of mathematics based on the notion of 'structure'.

This suggestion led Hellman ('Category Theory')—refining ideas articulated by Solomon Feferman—to argue that, as it stands, CT is incapable of serving as a foundational framework for articulating structuralism, because it depends on a prior background account of 'mapping' or 'relation' of the type provided by, e.g., set theory or a higher-order logic. In so doing, Hellman revitalized a debate whose primary contemporary participants are himself ('Structuralism', 'Categorical Structuralism', 'Foundational Frameworks'), Colin McLarty ('Categorical Structuralism', 'Learning', 'Recent Debate'), who insists that specific category-based theories can provide such a framework, and Awodey ('Answer'), who maintains that no such foundation is required.[26]

As a prelude to this debate, note that there are two conceptions of mathematical axioms.[27] According to the first—labeled *assertory* (or *Fregean*[28])—axioms are basic truths. Importantly, the primitive terms of assertory axioms have determinate meanings, which contribute to the said axioms' truth. The axioms of so-called non-algebraic theories (e.g., arithmetic, Euclidean geometry and set theory) have, traditionally, been so understood. Alternatively, mathematical axioms can be understood as specifying *defining conditions* on the subject matter of interest. The primitive terms of axioms so conceived—variously labeled *algebraic*, *schematic*, or *Hilbertian*—fail to have determinate meaning. Rather, such terms only acquire a meaning in the context of a particular interpretation. Such an interpretation is said to *satisfy* the axioms in question. The axioms of so-called algebraic theories (e.g., group theory, ring theory and field theory) are standardly understood schematically.

A central thesis of Hellman ('Category Theory') is that the CT axioms are always understood schematically. Using it, Hellman argues that CT "is defective as a [foundational] framework for structuralism in at least two major interrelated ways: *it lacks an external theory of relations*, and *it lacks substantive axioms of mathematical existence*" (138). That the second defect might be a corollary of Hellman's thesis is easily seen; plausibly, in order for axioms to have implications concerning the existence of mathematical entities, they must be asserted. Thus, from Hellman's perspective, "the question [of mathematical existence] really just does not seem to be addressed [by CT]" (136). That one might take the first defect to be a consequence of the above thesis is less immediate. Hellman begins by noting that this thesis requires us to "make sense of … *structures satisfying* the axioms of category theory" (135). He continues:

> … it is at this level that an appeal to 'collection' and 'operation' in *some* form seems unavoidable. Indeed, one can subsume both these notions under a logic or theory of *relations* (with collections as unary relations): that is what is missing from category and topos[29] theory, *both* as first-order theories *and*, *crucially*, *as informal mathematics*, but is provided by set-theory.

To summarize, since CT is algebraic, we need to makes sense of structures *satisfying* its axioms, which in turn makes CT dependent on an external theory of relations.[30]

Awodey's ('Answer' 55) response is to maintain that Hellman "misses the point of my proposal, which is … to use category theory to avoid the whole business of 'foundations'". He (55) notes that Hellman's concern with foundations reflects his bottom-up perspective on mathematics, which emphasizes "content" and "construction" over "form" and "description". According to Awodey, those with a bottom-up perspective seek to construct specific mathematical objects within some foundational system that, by itself, includes enough objects to represent all mathematical objects and enough laws, rules, and axioms to warrant all of the inferences and arguments found within mathematics. By contrast, those with a top-down perspective seek to specify only the information relevant to the mathematical job at hand without any thought of the ultimate nature of the objects involved. They also recognize that the laws, rules, and axioms used in various mathematical contexts exhibit some variation. Awodey's specific top-down perspective, which he believes is best articulated using the framework provided by CT, has certain affinities to Russell's if-thenism. In particular, "[e]very mathematical theorem is of the form 'if such-and-such is the case, then so-and-so holds'" (58). Yet, Awodey claims that his perspective does not suffer the problems associated with if-thenism, for, according to it, "the truth of the consequent statement does not depend on some unknown or unknowable antecedent condition; rather it applies only to those cases specified by the antecedent description" (60). In other words, "like axiomatic definitions, [antecedent conditions] serve to specify the range of applicability of the subsequent statement" (60). Moreover, these antecedent conditions should not be spelled out, as Russell does, by means of "universal quantification in a particular foundational system" (59), because "treating the indeterminate objects involved as universally quantified variables" does not capture the "'schematic' element of mathematical theorems, definitions, and … proofs" (59). Unfortunately, there is not sufficient space to further articulate or evaluate Awodey's subtle and interesting perspective here.

McLarty ('Categorical Structuralism') observes that CT is not offered as a categorical foundation for mathematics. Rather, those who promote categorical foundations ascribe a foundational role to such theories as ETCS—Lawvere's ('Elementary Theory') *Elementary Theory of the Category of Sets*—and CCAF—the *Category of Categories As Foundation* (see Lawvere ('Category of Categories'), and, relatedly, McLarty ('Category of Categories')). The axioms of ETCS and CCAF provide a category-theoretic description of their subject matter[31]—respectively, sets and functions between sets, and categories and functors between categories. Moreover, McLarty ('Categorical Structuralism' 45) maintains that the axioms of ETCS and CCAF can be asserted. Thus, Hellman's ('Category Theory') argument that these foundational frameworks are dependent on an external theory of relations is ineffective. Further, ECTS and CCAF offer specific, though different, answers to questions concerning which mathematical entities exist. Consequently, Hellman's ('Category Theory' 138) claim that categorical foundational frameworks "*lack substantive axioms of mathematical existence*" is mistaken—see, McLarty ('Categorical Structuralism' 42).[32]

Hellman ('Categorical Structuralism') offers different thoughts about the legitimacy of ETCS and CCAF as foundational frameworks. According to him, ETCS's primary fault is its failure to adequately account for the open-endedness of mathematics, i.e., the apparent indefinite extendability of mathematics. In particular, Hellman (154) takes ETCS to be committed to a "convenient fiction", i.e., "a fixed, presumably maximal, real-world universe of sets, '*the* category of sets'".[33]

Hellman's concerns about CCAF are best understood by considering the general constraints that he takes to govern foundational frameworks, specifically, the condition that the primitive concepts and axioms of such a framework should be intelligible in light of ordinary, mathematical and scientific experience.[34] Hellman believes that, because the CCAF axioms have as their intended interpretation a category of categories and functors, 'category' and 'functor' are among its primitives. Consequently, he consults mathematical practice, which leads him to the conclusion that the only conceptual access afforded to the primitive 'category' that is capable of sustaining a theory of the category of categories that is *sufficiently general* for that theory to serve as an appropriate foundation for mathematics is provided by characterizations of the type "[structure] … satisfying the algebraic axioms of CT" ('Categorical Structuralism' 157). Hellman recognizes that there are numerous examples within mathematical practice that illustrate the primitives 'category' and 'functor' and thus afford a certain type of conceptual access to them. Yet, from his perspective, this type of access is simply not sufficient to allow for appropriate generalization; in so generalizing, we must rely on characterizations of the type quoted above. Moreover, all such characterizations invoke satisfaction or some other notion of acceptability that depends on an external theory of relations.[35] Thus, according to Hellman, McLarty's intended interpretation of CCAF is, at a conceptual level, dependent on an external theory of relations.

McLarty rejects Hellman's conclusion. He takes the numerous illustrative examples of 'category' and 'functor' within mathematical practice to provide sufficient access to these concepts for the axioms of CCAF to be perfectly intelligible. Moreover, he takes Hellman's insistence on invoking the above types of characterizations to be an endorsement of a wide spread, but mistaken, belief that mathematical objects can only be defined in a fully intelligible way by means of a set-theoretic explication or something similar. For McLarty, mathematical practice outside of logic and set theory plainly demonstrates that mathematical objects are routinely defined in perfectly intelligible ways without the benefits of such explications—see, e.g., McLarty ('Learning').[36]

Clearly, Hellman and McLarty have radically divergent perspectives on categorical foundations. It seems likely that the debate between them will persist well into the future.

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[1] Throughout, by structuralism and structuralist, I mean *mathematical* structuralism and *mathematical* structuralist.
[2] I shall use 'place' throughout this paper.
[3] For details concerning neutral structures and the need for their inclusion in this characterization of ES, see Hellman ('Mathematics without Numbers' 116-7, 120).
[4] See, e.g., Russell, Putnam ('Thesis', 'Mathematics without Foundations'), and Hellman's writings on modal structuralism.
[5] See, e.g., Bourbaki's work on set-theoretic structuralism.

[6] Reck and Price offers an informative discussion of ESs, which places them in historico-philosophical context. Unfortunately, it largely ignores the top-down conception of structure. Those interested in this and its impact on the classification of ESs should consult Landry and Marquis. Discussions of ES can also be found in Parsons' work, Hellman ('Three Varieties', 'Structuralism'), and Shapiro ('Philosophy of Mathematics').
[7] SGS is the standard variety of NES. So far as I am aware, only Parsons endorses NES without endorsing SGS. He writes "[i]f what the numbers are is determined only by the structure of numbers, it should not be part of the nature of numbers that none of them is identical to an object given independently" ('Structuralism and Metaphysics' 61).
[8] This conception is so labeled, because it is built up by abstraction from systems with non-structural properties.
[9] The impact of a top-down perspective on debates over the purported metaphysical insight of SGS is a topic in need of further exploration.
[10] A similar observation is defended by Linnebo ('Structuralism').
[11] The clearest expression of Dependence of which I am aware is provided by Shapiro ('Thinking' 258). See, also, the third quote from Resnik and the first quote from Shapiro below.
[12] See Linnebo ('Structuralism') for a discussion of the relationship between these theses.
[13] See Burgess, Hellman ('Three Varieties', 'Structuralism without Structures'), Linnebo, MacBride ('Structuralism'), and Shapiro ('Structure and Identity').
[14] Shapiro ('Structure and Identity' §1) seeks a true formulation of P-Incompleteness with only limited success, then offers a resolution to the problem presented by these counterexamples: SGSists "can simply give up [P-Incompleteness]" (120), for such a "concession [would] not affect the underlying philosophy of [SGS]" (121), which, increasingly, Shapiro ('Structure and Identity', 'Tale of *i* and –*i*') equates with a version of Dependence.
[15] I shall not address the debate over whether the properties of application of mathematical objects are part of their nature.
[16] Foreseeing the objection discussed in this paragraph, Resnik ('Science of Patterns') reinterprets the data on which it is based, i.e., the *prima facie* cross-structural identifications. MacBride ('Structuralism Reconsidered' §2) argues that Resnik's reinterpretation of these data is subject to a related conflict with mathematical practice.
[17] Aware of such cross-structural identifications, Shapiro ('Philosophy of Mathematics' §3.2) shies away from ID-Incompleteness. MacBride ('Structuralism Reconsidered' §3) argues that both Shapiro's ('Philosophy of Mathematics') and ('Structure and Identity') treatments of the issues raised by such identifications conflict with mathematical practice.
[18] See Burgess, Button, Hellman ('Three Varieties', 'Structuralism'), Keränen's work, and MacBride ('Structuralism Reconsidered').
[19] See Ketland, Ladyman ('Identity and Diversity'), Leitgeb and Ladyman, and Shapiro ('Structure and Identity', 'Tale of *i* and –*i*').
[20] Two objects are weakly discernible iff there is a two-place, irreflexive relation that they satisfy (e.g., *i* and –*i* are weakly discernible in virtue of satisfying the relation 'is the additive inverse of'.)
[21] The nodes of certain unlabeled graphs are not weakly discernible.
[22] I advocate reformulating SGS to align it with the social-institutional account of mathematics endorsed in Cole. A manuscript articulating this reformulation of SGS and arguing that it can overcome the problems discussed in this section is under preparation.
[23] See Awodey ('Answer') and Landry and Marquis for discussion of the difference between these two conceptions.
[24] See, e.g., Awodey ('Structuralism' §1), Landry and Marquis, and McLarty ('Achieves', 'No Ontology').
[25] 'Arrow-theoretic' refers to the top-down approach to mathematics via structure-preserving mappings, which, in category theory, are called *morphisms* and represented by arrows.
[26] Landry and Shapiro ('Categories') also make valuable contributions to this debate, but these are slightly tangential to the aspects on which we shall concentrate.
[27] See Landry and Marquis and Shapiro ('Categories') for discussion of these conceptions beyond that found in the works cited in the previous paragraph.
[28] This is the traditional conception of axioms. It goes back at least as far as Euclid. It is labeled Fregean, because Frege invoked it in his famous correspondence with Hilbert, who, in the course of their correspondence, explicitly articulated the second conception of axioms, which arose during the development of abstract algebra.
[29] A topos is a type of category.
[30] It is informative to compare Hellman's claims with those relating to metamathematics in Shapiro ('Categories').

[31] For example, ETCS provides a description of sets that treats composition of functions, rather than set-membership, as primitive.
[32] Landry offers an interesting, slightly different interpretation of McLarty, informed by his (‘Learning’), which aligns McLarty more closely with her own view.
[33] McLarty (‘Recent Debate’) claims: “This is not my fiction”.
[34] These constraints are summarized in Hellman (‘Foundational Frameworks’ §2). This particular way of understanding desideratum 3 from that paper was suggested to me in a personal communication from Hellman on 11/20/2009.
[35] For more on this point, see Landry and Shapiro (‘Categories’).
[36] This way of understanding McLarty’s response to Hellman was suggested to me in a personal communication from McLarty on 11/22/2009.